



Complex Numbers

The set \mathbb{C} of complex numbers is a set larger than the set \mathbb{R} of real numbers, constructed as follows:

We first define i to be a number s.t. $i^2 = -1$.

Then, we define $\mathbb{C} := \{x+iy : x, y \in \mathbb{R}\}$

For any $z = x + iy \in \mathbb{C}$, we call
 $x \in \mathbb{R}$ the real part of z , $\text{Re } z$,

$y \in \mathbb{R}$ the imaginary part of z , $\text{Im } z$.

We define addition and multiplication on \mathbb{C} as follows:

If $x_1, y_1, x_2, y_2 \in \mathbb{R}$, let:

- $(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i \cdot (y_1+y_2)$, and

- $(x_1+iy_1) \cdot (x_2+iy_2) = \underbrace{x_1x_2 + x_1 \cdot iy_2 + iy_1x_2}_{\substack{\text{for distributivity} \\ \text{to hold}}} + \overbrace{iy_1iy_2}^{= -y_1y_2} =$
 $= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

②

We demand that these operations have the usual properties of addition and multiplication on \mathbb{R} (commutativity, associativity, distributivity).

We also write $\boxed{\frac{1}{x+iy}}$ as the complex number that, multiplied with $x+iy$, gives 1.
(when non-zero)

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$, with $x_2+iy_2 \neq 0$, we have then:

$$\frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1) \cdot (x_2-iy_2)}{(x_2+iy_2) \cdot (x_2-iy_2)} = \frac{(x_1+iy_1) \cdot (x_2-iy_2)}{x_2^2 + y_2^2 \in \mathbb{R}}$$

→ ex: • $(5+3i) + (2+10i) = 7+13i$.
• $(5+3i) \cdot (2+10i) = 5 \cdot 2 + 5 \cdot 10i + 3i \cdot 2 + 3i \cdot 10i = 10 + 50i + 6i - 30 = -20 + 56i$.

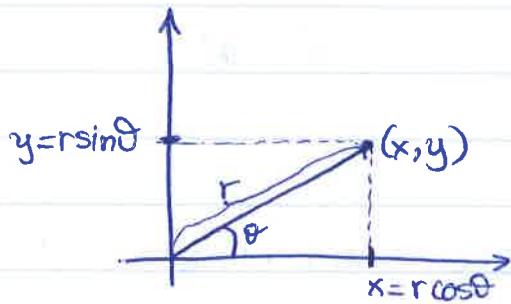
$$\bullet \frac{5+3i}{2+10i} = \frac{(5+3i) \cdot (2-10i)}{(2+10i)(2-10i)} = \frac{5 \cdot 2 - 5 \cdot 10i + 3i \cdot 2 - 3i \cdot 10i}{2^2 - (10i)^2} =$$

$$= \frac{10 - 50i + 6i + 30}{2^2 + 10^2} = \frac{40 - 44i}{104} = \frac{40}{104} - \frac{44}{104} \cdot i$$

(3)

→ We can represent each complex number on the plane:

Each complex number $x+iy$ ($x, y \in \mathbb{R}$) can be represented as the point (x, y) on the plane:



So, we can imagine $x+iy$ as the vector (x, y) .

Notice that, for any vector (x, y) , we have

that : $x = r \cos \theta$, where $r = \sqrt{x^2 + y^2}$
and $y = r \sin \theta$ is the length of

⚠ We can imagine the sum of two complex numbers as the sum of two vectors!

θ is the angle of the vector (x, y) with the horizontal half-line $[0, +\infty)$.

That is, for these r and θ , we have that

$$(x, y) = (r \cos \theta, r \sin \theta) = r \cdot (\cos \theta, \sin \theta), \text{ which means that}$$

$$x+iy = r \cos \theta + i r \sin \theta = r \cdot (\cos \theta + i \sin \theta).$$

(4)

Notice that $(\cos\theta, \sin\theta)$ is a unit vector, i.e. a

vector with length 1 (as its length equals $\sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$).

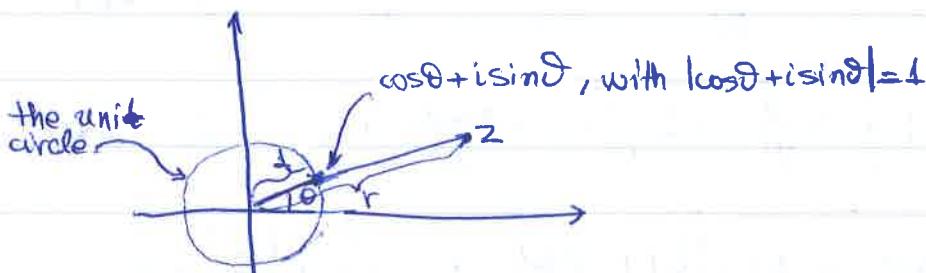
For any $z = x + iy \in \mathbb{C}$, we now define

$$|z| := \sqrt{x^2 + y^2} ; \text{ the length of } z \text{ when } z \text{ is seen as a vector in } \mathbb{R}^2.$$

We have thus shown that :

$$\text{If } z = x + iy \in \mathbb{C}, \quad z = r \cdot (\cos\theta + i\sin\theta),$$

where $r = |z|$, θ is the angle z creates with $[0, \text{too}]$,
and $\cos\theta + i\sin\theta$ is the unit vector with the same direction as z :



(5)

Let us now take this a little further :

Remember that

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

and $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$

Now, if it made sense to write

$$i \cdot \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right) = i\theta - i \cdot \frac{\theta^3}{3!} + i \cdot \frac{\theta^5}{5!} - i \cdot \frac{\theta^7}{7!} + \dots$$

(which we don't know makes sense yet,
since we haven't defined an infinite sum
of complex numbers),

we would have:

$$i \sin \theta = (i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \dots, \text{ and}$$

$$\cos \theta = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \dots, \text{ so that}$$

$$\cos \theta + i \sin \theta = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

again, if this made sense $= e^{i\theta}$. This motivates the study of:

(6)

→ **Complex series:**
(part 1)

We have defined an infinite sum of real numbers to be the limit of the partial sums of the infinite sum.

However, in the case of a complex series $\sum_{k=0}^{+\infty} a_k \in \mathbb{C}$,

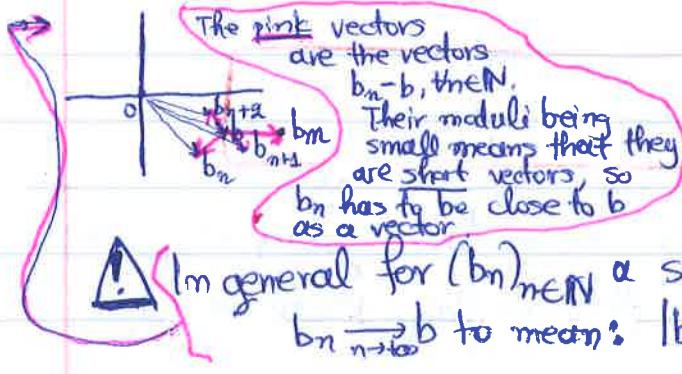
we are actually summing infinitely many vectors. So, it makes sense that the infinite sum should be a complex number (another vector). We define it as such:

→ **Def:** Let $\sum_{k=1}^{+\infty} a_k$ be a complex series (i.e., $a_k \in \mathbb{C} \forall k \in \mathbb{N}$)

We say that the series converges to some $a \in \mathbb{C}$,

and we write $\sum_{k=1}^{+\infty} a_k = a$, if
 $s_n := a_1 + a_2 + \dots + a_n$ $\xrightarrow{\text{as } n \rightarrow +\infty} a$

which in turn means that $|s_n - a| \xrightarrow{n \rightarrow +\infty} 0$.



i.e., the length of the vector $s_n - a (\in \mathbb{C})$ goes to 0 as $n \rightarrow +\infty$

⚠ In general for $(b_n)_{n \in \mathbb{N}}$ a sequence of complex numbers, we define $b_n \xrightarrow{n \rightarrow +\infty} b$ to mean: $|b_n - b| \xrightarrow{n \rightarrow +\infty} 0$. If $b_n \in \mathbb{R}$ then this coincides with the definition of limits in \mathbb{R} .

(F)

→ Observation: Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. In particular, let

$$b_n = x_n + i y_n, \quad x_n, y_n \in \mathbb{R}, \quad n \in \mathbb{N}.$$

We have said that the definition of $b_n \xrightarrow[n \rightarrow \infty]{\mathbb{C}} b \in \mathbb{C}$

is $|b_n - b| \xrightarrow[n \rightarrow \infty]{} 0$, i.e. $\sqrt{(x_n - x)^2 + (y_n - y)^2} \xrightarrow[n \rightarrow \infty]{} 0$.

One can easily show that this is equivalent to

$x_n \rightarrow x$
and $y_n \rightarrow y$ as $n \rightarrow \infty$.

So: $(b_n)_{n \in \mathbb{N}}$ converges

to b iff its real parts converge to $\operatorname{Re} b$
and its imaginary parts converge to $\operatorname{Im} b$.

→ Corollary: Consider the complex series $\sum_{k=1}^{\infty} (a_k + i b_k)$.

Then, this series converges iff $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge.

In that case, $\sum_{k=1}^{\infty} (a_k + i b_k) = \sum_{k=1}^{\infty} a_k + i \sum_{k=1}^{\infty} b_k$

Proof: $\left(n\text{-th partial sum for } \sum_{k=1}^{\infty} (a_k + i b_k) \right) = (a_1 + i b_1) + \dots + (a_n + i b_n) = (a_1 + \dots + a_n) + i (b_1 + \dots + b_n)$. Since the real part converges

(to the sum $\sum_{k=1}^{+\infty} a_k$) and the imaginary part converges
 (to the sum $\sum_{k=1}^{+\infty} b_k$), we have that the whole thing
 converges to $\sum_{k=1}^{+\infty} a_k + i \cdot \sum_{k=1}^{+\infty} b_k$. ⑧ ■

More generally:

For complex series, we have the usual properties as
 for real series:

If $\sum_{k=1}^{+\infty} a_k$ and $\sum_{k=1}^{+\infty} b_k$ are convergent series,

then :

$$\textcircled{*}_1 \quad \sum_{k=1}^{+\infty} (a_k + b_k) = \sum_{k=1}^{+\infty} a_k + \sum_{k=1}^{+\infty} b_k, \quad \text{and}$$

\curvearrowright converges

$$\textcircled{*}_2 \quad \sum_{k=1}^{+\infty} (\lambda a_k) = \lambda \cdot \sum_{k=1}^{+\infty} a_k, \quad \forall \lambda \in \mathbb{C}.$$

\curvearrowright converges

Proof: Let

$$s_n = a_1 + \dots + a_n$$

$$\text{and } s'_n = b_1 + \dots + b_n.$$

Since $\sum_{k=1}^{+\infty} a_k$, $\sum_{k=1}^{+\infty} b_k$ are convergent, we have

$$s_n \rightarrow s = \sum_{k=1}^{+\infty} a_k$$

$$\text{and } s'_n \rightarrow s' = \sum_{k=1}^{+\infty} b_k$$

the n-th partial sum for $\sum_{k=1}^{+\infty} (a_k + b_k)$

$$\text{So: } \underbrace{s_n + s'_n}_{\text{n-th partial sum for } \sum_{k=1}^{+\infty} (a_k + b_k)} \rightarrow s + s',$$

i.e. $\sum_{k=1}^{+\infty} (a_k + b_k) = \sum_{k=1}^{+\infty} a_k + \sum_{k=1}^{+\infty} b_k$

Moreover, the n -th partial sum for $\sum_{k=1}^{+\infty} (\lambda a_k)$ is

$$(\lambda a_1) + (\lambda a_2) + \dots + (\lambda a_n) = \lambda \cdot (a_1 + a_2 + \dots + a_n) = \lambda \cdot S_n \xrightarrow{n \rightarrow \infty} \lambda \cdot s$$



In the proof above, we used that, for any

complex sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, with

$$\underline{a_n \rightarrow a \in \mathbb{C}} \quad \text{and} \quad \underline{b_n \rightarrow b \in \mathbb{C}} \quad \text{as } n \rightarrow \infty,$$

and for any $\lambda \in \mathbb{C}$, we have:

$$\underline{a_n + b_n \xrightarrow{n \rightarrow \infty} a + b}$$

$$\text{and } \underline{\lambda a_n \xrightarrow{n \rightarrow \infty} \lambda a}.$$

→ it can also be proved using the definition, as long as we prove first, for instance, that $|z_1 z_2| = |z_1| |z_2|$ if $z_1, z_2 \in \mathbb{C}$ (required).

This can be proved using the observation earlier, that convergence for a complex sequence is equivalent

to convergence for the sequence of its real parts
and the sequence of its imaginary parts

→ Expressing $\cos\theta + i\sin\theta$ as $e^{i\theta}$:

(10)

Let us now make a pause, to get back to our musings about expressing $\cos\theta + i\sin\theta$ as $e^{i\theta}$; let us explain this. By the properties of series we have described so far, we know that:

$$i \cdot \sin\theta = i \cdot \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \text{ by } (4)$$

$$= (i\theta) - i \cdot \frac{\theta^3}{3!} + i \cdot \frac{\theta^5}{5!} - i \cdot \frac{\theta^7}{7!} + \dots \quad \begin{matrix} \text{check it! We have} \\ (i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, \\ i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1) \end{matrix}$$

$$= (i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \dots \quad \text{HDER,}$$

$$\text{while } \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots =$$

$$= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \dots, \quad \text{HDER.}$$

So, since the above complex series converge, by (4) their sum converges, and equals the series with (k -th term) = (k -th term of $i\sin\theta$) + (k -th term of $\cos\theta$),

i.e. the series $1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$ We have

just shown that this series converges, and equals $\cos\theta + i\sin\theta$! We see also that it is a generalisation of the series expansion of e^x , for $x = i\theta$.

(11)

So, that prompts us to formally define

$$e^z := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots, \quad \forall z \in \mathbb{C};$$

and so far we know that this series converges for $z \in \mathbb{R}$ and $z = i\theta$, for $\theta \in \mathbb{R}$.

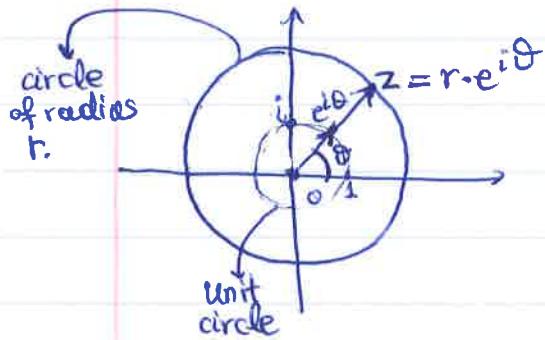
(We will later show that e^z converges $\forall z \in \mathbb{C}$).

With this definition of e^z , we have that

$$\cos \theta + i \sin \theta = e^{i\theta}, \quad \forall \theta \in \mathbb{R} \quad \boxed{\text{Euler's formula}}$$

a unit vector,
 $\forall \theta \in \mathbb{R}$.

Thus, we know that, $\forall z \in \mathbb{C}$, with $z = x + iy$,



$$z = r \cdot (\cos \theta + i \sin \theta) = r \cdot e^{i\theta},$$

where $r = \sqrt{x^2 + y^2}$, the length of the vector (x, y) ,

and θ is the angle the vector z creates with the horizontal half-line $[0, \text{too})$.

So, $e^{i\theta}$ is the unit vector with the same direction as z .

Lecture 8:

12 Sep 2016.

1

Complex series :

(part 2)

→ apart from testing for convergence
the real part and the
imaginary part of
the series, which
we have already
discussed.

To test complex series for convergence, we use two facts:

Preliminary test

$$\text{If } \sum_{k=1}^{\infty} a_k \text{ converges, then } a_k \xrightarrow{k \rightarrow \infty} 0$$

converges, then

$$a_k \xrightarrow{k \rightarrow \infty} 0$$

by definition:
equivalent to $\lim_{k \rightarrow \infty} |a_k| = 0$

(this implies that, if $a_k \not\rightarrow 0$, then $\sum_{k=1}^{\infty} a_k$ doesn't converge).
i.e., $|a_k| \not\rightarrow 0$

Proof: Like for real series (exercise; remember what a limit of complex numbers is). ■

→ Let $\sum_{k=1}^{\infty} a_k$ be a complex series. If it converges absolutely (i.e. if $\sum_{k=1}^{\infty} |a_k|$ converges), then it converges.

(2)

Already these two facts tell us that, to test a

complex series

$\sum_{k=1}^{+\infty} \alpha_k$ for convergence, we can use:

- the preliminary test for $\sum_{k=1}^{+\infty} |\alpha_k|$, and
- any test for convergence of the real series

$\sum_{k=1}^{+\infty} |\alpha_k|$ (simple and limiting comparison tests, ratio and root test). If $\sum_{k=1}^{+\infty} |\alpha_k|$ converges,

then $\sum_{k=1}^{+\infty} \alpha_k$ converges too. But if $\sum_{k=1}^{+\infty} |\alpha_k|$ diverges,

then we don't necessarily have that $\sum_{k=1}^{+\infty} \alpha_k$ diverges.

for instance, we need to think before saying that, if the ratio / root / tests imply divergence for $\sum_{k=1}^{+\infty} |\alpha_k|$, limiting comparison

then we also have divergence for $\sum_{k=1}^{+\infty} \alpha_k$. Thankfully

though, for the ratio and root tests, this is true:

→ **Ratio test** for complex series: Let $\sum_{k=1}^{+\infty} \alpha_k$ be a complex series, with $\alpha_k \neq 0 \ \forall k \in \mathbb{N}$. 3.

Suppose that $\lim_{k \rightarrow \infty} \frac{|\alpha_{k+1}|}{|\alpha_k|} = l$ exists in $\mathbb{R} \cup \{+\infty\}$.

Then:

- If $l < 1$, then $\sum_{k=1}^{+\infty} |\alpha_k|$ converges ($\Rightarrow \sum_{k=1}^{+\infty} \alpha_k$ converges) as before, for real series (compare with a geometric series)
- If $l > 1$, then $\sum_{k=1}^{+\infty} |\alpha_k|$ diverges. as for real series (show that $\alpha_k \not\rightarrow 0$)
- If $l = 1$, the test is inconclusive.

→ **Root test** for complex series: Let $\sum_{k=1}^{+\infty} \alpha_k$ be a complex series.

Suppose that $\lim_{k \rightarrow \infty} \sqrt[k]{|\alpha_k|} = l$ exists in $\mathbb{R} \cup \{+\infty\}$.

Then:

- If $l < 1$, then $\sum_{k=1}^{+\infty} |\alpha_k|$ converges ($\Rightarrow \sum_{k=1}^{+\infty} \alpha_k$ converges).
- If $l > 1$, then $\sum_{k=1}^{+\infty} |\alpha_k|$ diverges.
- If $l = 1$, then the test is inconclusive.

(4)

To sum up: to test convergence for $\sum_{k=1}^{\infty} a_k z^k$, we can:

- use the preliminary test for $\sum_{k=1}^{\infty} |a_k|$.
- use comparison tests for $\sum_{k=1}^{\infty} |a_k|$.
- use the ratio / root tests for $\sum_{k=1}^{\infty} a_k$.

Now that we know what it means for a complex series to converge, we can talk about:



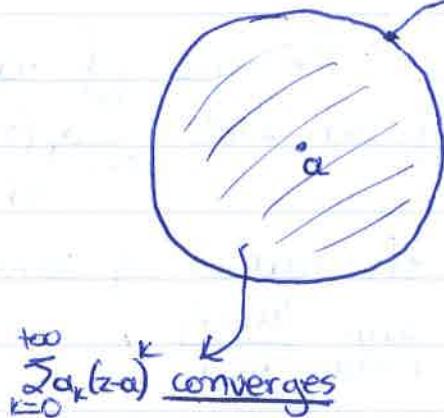
Complex power series:

A complex power series centered at $a \in \mathbb{C}$ is a series of the form $\sum_{k=0}^{\infty} a_k (z-a)^k$. For such a series, we always have that: there exists some $R \geq 0$, s.t.

- $\sum_{k=0}^{\infty} a_k (z-a)^k$ converges when $|z-a| < R$ (i.e., inside the open disc $D(a, R)$)
- $\sum_{k=0}^{\infty} a_k (z-a)^k$ diverges when $|z-a| > R$ (i.e., outside the disc $D(a, R)$ and its boundary $\partial D(a, R)$).
- $\sum_{k=0}^{\infty} a_k (z-a)^k$ may have special behaviour if $z \in \partial D(a, R)$.

(5)

I.e., the following picture holds: for $\sum_{k=0}^{\infty} a_k(z-a)^k$, there exists some $R \geq 0$ (maybe $R = \infty$), s.t.



anything can happen
for $z \in \partial D(a, R)$, the circle.

$\sum_{k=0}^{\infty} a_k(z-a)^k$ diverges
for z outside the disc and its boundary,

i.e. in

$$\mathbb{C} \setminus (D(a, R) \cup \partial D(a, R))$$

 $= \{z \in \mathbb{C} : |z-a| > R\}.$

for z inside the open disc $D(a, R) = \{z \in \mathbb{C} : |z-a| \leq R\}$.

if $R = \infty$,
this is the whole of \mathbb{C} .

Notice that $\sum_{k=0}^{\infty} a_k(z-a)^k$ always converges for $z=a$ (to a_0).

We call this R the radius of convergence of $\sum_{k=0}^{\infty} a_k(z-a)^k$,

and $D(a, R)$ the disc of convergence of $\sum_{k=0}^{\infty} a_k(z-a)^k$.

(34)



Since the **ratio** and **root** tests can be applied for complex series, they can also be applied for complex **power** series.

Try them; just like in the case of real power series, for a complex power series $\sum_{k=0}^{\infty} a_k(z-a)^k$:

the ratio test gives the radius of convergence to be $\frac{1}{l}$, for $l = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}$,

as long as this limit exists
and $a_k \neq 0 \forall k \in \mathbb{N}$,

while

the root test gives the radius of convergence to be $\frac{1}{l}$, for $l = \lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}$,

as long as this limit exists.

Using, for instance, the ratio test, one can show that, if we formally define

$$\cos z := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad \forall z \in \mathbb{C},$$

and

$$\sin z := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \quad \forall z \in \mathbb{C},$$

(1)

then the above series converge $\forall z \in \mathbb{C}$. This way, we have extended the cosine and sine functions from \mathbb{R} to the whole of \mathbb{C} .

It is easy to check that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \forall z \in \mathbb{C},$$

$$\text{and } \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \forall z \in \mathbb{C};$$

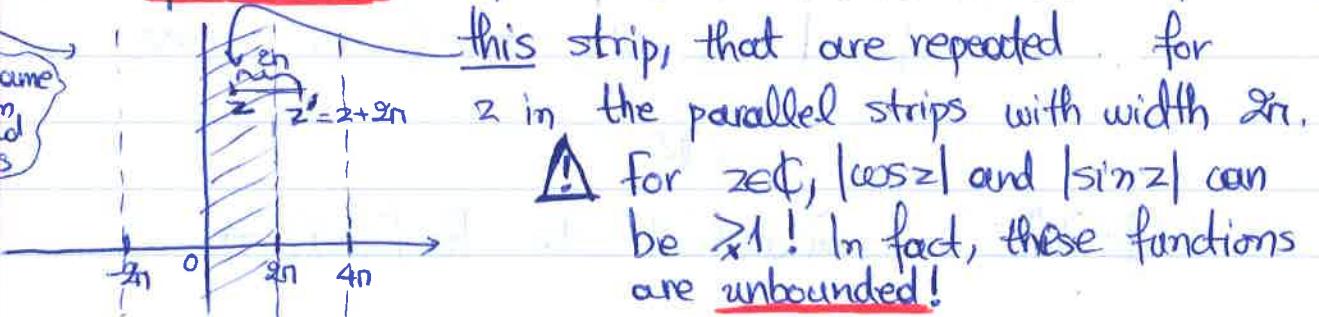
this is because the $(n\text{-th partial sum of } \cos z)$

$$\text{equals } \frac{(\text{the } n\text{-th partial sum of } e^{iz}) + (\text{the } n\text{-th partial sum of } e^{-iz})}{2},$$

and similarly for $\sin z$.

Using $\textcircled{*}$, we can see that \cos and \sin are periodic, with period $2\pi i$: so, $\cos z$ and $\sin z$ take values for z in

z, z'
have same
 \sin and
 \cos



⚠ for $z \in \mathbb{C}$, $|\cos z|$ and $|\sin z|$ can be ≥ 1 ! In fact, these functions are unbounded!

(2)

→ We can easily see that the geometric

series $\sum_{k=0}^{\infty} z^k$ converges $\iff |z| < 1$.

In particular (following the same proof as for z real),

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \text{ for } |z| < 1.$$



functions of a complex variable

→ Def: Let $f: D(z_0, r) \rightarrow \mathbb{C}$

↳ some open disc centered at z_0

We say that $\lim_{z \rightarrow z_0} f(z) = w \in \mathbb{C}$

if $|f(z) - w| \xrightarrow{\epsilon \in \mathbb{R}} 0$ as $z \rightarrow z_0$.

(3)

If we denote $z = x + iy \in \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \mathbb{C}$,

then this means that

$$\operatorname{Re}(f(x,y) - w) \rightarrow 0 \text{ and } \operatorname{Im}(f(x,y) - w) \rightarrow 0 \quad (\text{i.e., } x \rightarrow x_0 \text{ and } y \rightarrow y_0)$$

(I.e.: $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $(x,y) \in D(z_0, \delta)$, then $f(x,y) \in D(w, \epsilon)$)



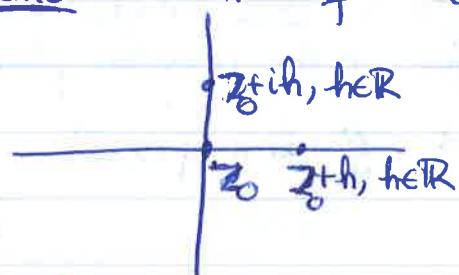
$$(x, y) \rightarrow (x_0, y_0)$$

$$f(x, y) \in D(w, \epsilon)$$

This means that, no matter how we approach z_0 with complex numbers z on the plane, $f(z)$ will always approach w .



It may be that we get the same limit if we approach z_0 horizontally (i.e. with z of the form $z_0 + ih$, $h \in \mathbb{R}$),



as when we approach it vertically (i.e., with z of the form $z_0 + ih$, $h \in \mathbb{R}$).

I.e., we may have $\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(z_0 + h) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(z_0 + ih)$;

but this still doesn't mean that $\lim_{\substack{z \rightarrow z_0 \\ z \in \mathbb{C}}} f(z)$ exists!

(4)

We need to take into account all possible ways to approach z_0 on the complex plane!

→ Def: Let $f: \underbrace{D(z_0, r)}_{\substack{\text{some open} \\ \text{disc centered} \\ \text{at } z_0}} \rightarrow \mathbb{C}$. We say that

\downarrow
some open
disc centered
at z_0

f is continuous at z_0 if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

i.e., again, we want this to happen no matter how we approach z_0 on the complex plane!

So, again, even if $\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(z_0 + h) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(z_0 + ih) = f(z_0)$

(i.e. $\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(x_0 + h, y_0) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} f(x_0, y_0 + h) = f(x_0, y_0)$)

we still don't necessarily have that f is continuous at 0. (find an example of this!)

(5)

→ Def: Let $f: D(z_0, r) \rightarrow \mathbb{C}$.

If $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, we say that
 $\exists f$ for some $r > 0$

f is holomorphic at z_0 , and we define

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

After the above discussion, it may come as a surprise that the above limit as $z \rightarrow z_0$ exists

if and only if the limits as we approach z_0 horizontally and vertically exist and are equal!

All this is formulated in the theorem that follows.
 We will only prove the (\Rightarrow) direction.

(6)

→ Thm: Let $f: D(z_0, r) \rightarrow \mathbb{C}$, for some $z_0 = x_0 + iy_0 \in \mathbb{C}$
 i.e., f doesn't have to be defined on the whole of \mathbb{C} . and some $r > 0$.

We write $f(x+iy) = u(x, y) + iv(x, y)$, $\forall x, y \in \mathbb{R}$.

Then: f is holomorphic at z_0



all the partial derivatives $\frac{\partial u}{\partial x}(x_0, y_0)$, $\frac{\partial u}{\partial y}(x_0, y_0)$,
 $\frac{\partial v}{\partial x}(x_0, y_0)$, $\frac{\partial v}{\partial y}(x_0, y_0)$
exist,

and

and

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

Cauchy - Riemann
conditions



Notice that mere existence of the partial derivatives of the real and imaginary parts of f is not enough to make f differentiable in \mathbb{C} !

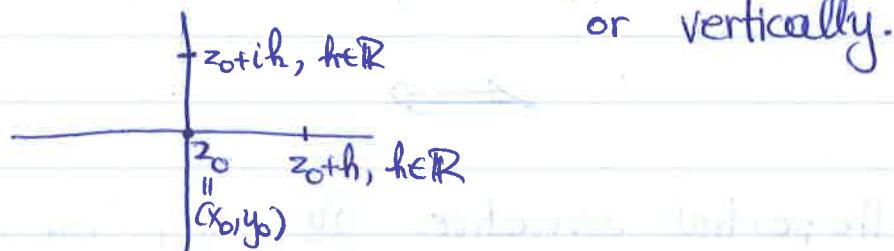
⑦

Proof (of \Rightarrow only) :

If $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, then we

get the same limit no matter how we approach z_0 ;

In particular, whether we approach it horizontally or vertically.



In particular, the limits

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{and}$$

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + ih) - f(z_0)}{ih}$$

exist and are equal (to $f'(z_0)$). And:

$$f'(z_0) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} =$$

$$= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{(u(x_0 + h, y_0) - u(x_0, y_0)) + i(v(x_0 + h, y_0) - v(x_0, y_0))}{h}$$

(8)

So:

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}, \quad \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}$$

both exist (both the real and imaginary part have to converge for the whole thing to converge),

i.e. $\frac{\partial u}{\partial x}(x_0, y_0)$, $\frac{\partial v}{\partial x}(x_0, y_0)$ exist, and

$$\boxed{\frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0) = f'(z_0)} \quad \textcircled{*1}$$

remember this!

Similarly, $\frac{\partial u}{\partial y}(x_0, y_0)$, $\frac{\partial v}{\partial y}(x_0, y_0)$ both exist, and

$$\frac{\partial v}{\partial y}(x_0, y_0) - i \cdot \frac{\partial u}{\partial y}(x_0, y_0) = \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0).$$

By $\textcircled{*1}$, $\textcircled{*2}$: $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$

and $\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$.

(9)

We will not prove the (\Leftarrow) direction. Notice, however, that, by the above,

$$\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial u}{\partial y}(x_0, y_0), \frac{\partial v}{\partial x}(x_0, y_0), \frac{\partial v}{\partial y}(x_0, y_0)$$

exist iff the limits

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{and} \quad \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{R}}} \frac{f(z_0+ih) - f(z_0)}{h}$$

$\underbrace{\qquad\qquad\qquad}_{l_1}$ $\underbrace{\qquad\qquad\qquad}_{l_2}$

and the Cauchy-Riemann conditions are satisfied

$$\text{iff } l_1 = l_2.$$

So, the (\Leftarrow) direction really says that:

If l_1, l_2 exist and $l_1 = l_2$ (i.e., if we get the same limit in C whether we approach z_0 horizontally or vertically),

then $f'(z_0)$ exists (and equals $l_1 (= l_2)$) (i.e., we

(10)

get the same limit no matter how we approach z_0).



→ Examples:

- $f(z) = z$ is holomorphic $\forall z \in \mathbb{C}$:

$$f(x+iy) = x+iy \quad , \text{ and} \\ u(x,y) = v(x,y)$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} (=1)$ on the whole of \mathbb{R}^2 , and

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} (=0)$ on the whole of \mathbb{R}^2 .

- $f(z) = \bar{z}$ is not holomorphic anywhere:

$$f(x+iy) = x-iy \quad , \text{ so}$$

$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$, on the whole of \mathbb{R}^2 .

(11)

→ Prop: Let f, g be holomorphic at $z_0 \in \mathbb{C}$, and let $\lambda \in \mathbb{C}$.

Then:

- $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.
- $(\lambda f)'(z_0) = \lambda \cdot f'(z_0)$.
- $(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$.
- $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0) \cdot g(z_0) - f(z_0) \cdot g'(z_0)}{g^2(z_0)}$, when $g(z_0) \neq 0$.
- $(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$, as long as $f'(g(z_0))$ exists.

Proof: Either with the definition of the derivative at z_0 ,

or by verifying the Cauchy-Riemann conditions and using the usual rules of partial differentiation in \mathbb{R}

(together with $f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0)$).

(12)

→ Examples:

- $(z^n)' = n \cdot z^{n-1}$, $\forall n \in \mathbb{N}$ (use product rule).
- $\left(\frac{1}{z}\right)' = -\frac{1}{z^2}$, $\forall z \neq 0$.
- $\left(\frac{1}{z^n}\right)' = -\frac{1}{(z^n)^2} \cdot (z^n)', \forall n \in \mathbb{N}, \forall z \neq 0$ (just like for $z \in \mathbb{R}$).

→ Def: $f: \mathbb{C} \rightarrow \mathbb{C}$ is called analytic at $z_0 \in \mathbb{C}$

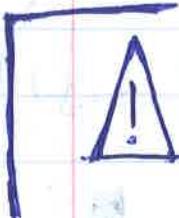
↓
or maybe
defined on
a subset of \mathbb{C}

if f can be written as
a power series around z_0 , i.e.:

if
$$f(z) = \sum_{k=0}^{+\infty} a_k (z-z_0)^k, \quad \forall z \in D(z_0, r),$$

for some $r > 0$.

(it doesn't matter which).



We have discussed that real functions don't have to be analytic, even if they are smooth. However, we will show that, if f is holomorphic at $z_0 \in \mathbb{C}$ (i.e. differentiable once at z_0 in \mathbb{C}), then f analytic at z_0 !

(13)

And, in particular, f will be infinitely many times differentiable at z_0 . On the way, some very useful results will come up.

→ Def: Let $f: \underbrace{[a,b]}_{\text{in } \mathbb{R}} \rightarrow \mathbb{C}$ continuous.

We define

$$\int_a^b \underbrace{f(t)}_{t \in \mathbb{C}} dt := \int_a^b \operatorname{Re}(f(t)) dt + i \cdot \int_a^b \operatorname{Im}(f(t)) dt.$$

this means that, when I draw the values $f(x)$ on the complex plane, as x runs from a to b , I don't lift my pencil from the paper!



ex: Let $f(t) = \sin t + it$, $t \in [0, 2\pi]$. Then:

$$\int_0^{2\pi} f(t) dt = \int_0^{2\pi} \sin t dt + i \cdot \int_0^{2\pi} t dt = 0 + i \cdot \left[\frac{t^2}{2} \right]_0^{2\pi} = 2 \cdot \pi^2.$$

→ Prop: Let $f, g: [a,b] \rightarrow \mathbb{C}$ continuous, $\lambda \in \mathbb{C}$. Then:

- $\int_a^b (\underbrace{f(t) + g(t)}_{t \in \mathbb{C}}) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$.
- $\int_a^b \underbrace{\lambda \cdot f(t)}_{t \in \mathbb{C}} dt = \lambda \cdot \int_a^b f(t) dt$.